Modeling macroeconomic series with regime-switching models characterized by a high-dimensional state space

Maciej Augustyniak[a,b] and Arnaud Dufays[*,c,d]

[a] *Département de mathématiques et de statistique, Université de Montréal, PO Box 6128, Station Centre-ville, Montreal, Quebec, Canada, H3C 3J7*

[b] *Quantact Actuarial and Financial Mathematics Laboratory, Centre de recherches mathématiques, Université de Montréal, PO Box 6128, Station Centre-ville, Montreal, Quebec, Canada, H3C 3J7*

[c] *Département d'économique, Université Laval, 1025 avenue des Sciences-Humaines, Quebec City, Quebec, Canada, G1V 0A6*

[d] *CRREP research associate*

Version: June 15, 2018

Abstract

The Markov-switching multifractal process, and recent extensions such as the factorial hidden Markov volatility model, correspond to tightly parametrized hidden Markov models characterized by a high-dimensional state space. Because the central component in these models is a Markov chain restricted to have positive support, the applicability of such models has been so far limited to the modeling of positive processes such as volatilities, inter-trade durations and trading volumes. By adapting the factorial hidden Markov volatility model, we develop a new regime-switching process for capturing time variation in the conditional mean of a time series with support on the whole real line. We show its promising performance to fit 21 widely used macroeconomic data sets.

JEL classification: C22, C51, C58

Keywords: conditional mean model, Markov-switching, factorial hidden Markov model, multifractal

Maciej Augustyniak acknowledges financial support from the Natural Sciences and Engineering Research Council of Canada. Arnaud Dufays acknowledges financial support from the Fonds de recherche du Québec – Société et culture and from the Social Sciences and Humanities Research Council of Canada via project 430-2017-00215.

[*] Corresponding author.
Email addresses: augusty@dms.umontreal.ca (Maciej Augustyniak), arnaud.dufays@ecn.ulaval.ca (Arnaud Dufays).

1 Introduction

The Markov-switching multifractal (MSM) model of Calvet and Fisher (2004) and subsequent inspired processes, such as the component-driven regime-switching model of Fleming and Kirby (2013) and the factorial hidden Markov volatility (FHMV) model of Augustyniak et al. (2018), have proven to be strong competitors to the GARCH family of models for fitting and forecasting volatilities of financial assets. Moreover, when proposed to analyze trading volumes (Lux and Kaizoji, 2007) and inter-trade durations (Chen et al., 2013), the MSM process proved once again to be an excellent alternative to standard models in the corresponding literatures.

The key ingredient making the MSM and related processes successful is a high-dimensional Markov chain that has the capability to produce a slowly decaying autocorrelation function which, on a finite interval, can mimic the hyperbolic decline generated by long memory processes (see Calvet and Fisher, 2004; Liu et al., 2007). However, this Markov chain is restricted to have positive support, which limits its applicability to model processes defined on the whole real line.

The objective of this letter is to develop a new regime-switching process, named the factorial hidden Markov mean (FHMM) model, for capturing time variation in the conditional mean of a time series with support on the real line. This process, inspired from the FHMV model proposed by Augustyniak et al. (2018), corresponds to a parsimoniously parametrized hidden Markov model characterized by a high-dimensional state space. Furthermore, we derive its autocovariance structure and show its promising performance to fit 21 widely used macroeconomic data sets.

2 Model definition and properties

Let y_t, $t = 1, \ldots, T$, denoted by $\{y_t\}$, represent a time series with support on the real line. The FHMM model is specified as follows:

$$\begin{aligned} y_t &= U_t + \epsilon_t, \\ U_t &= M_t(\mu + \alpha(C_t - 1)), \end{aligned} \tag{1}$$

where $\{\epsilon_t\}$ is an independent and identically distributed (i.i.d.) innovation process with mean 0 and variance σ^2, and $\{U_t\}$ represents an unobserved Markov chain which is assumed independent of $\{\epsilon_t\}$. The process $\{U_t\}$ plays the role of a dynamic conditional mean and its building blocks are the processes $\{C_t\}$ and $\{M_t\}$, where $\{C_t\}$ is a latent Markov chain satisfying $\mathbb{E}(C_t) = 1$, and $\{M_t\}$ is a sequence of i.i.d. discrete random variables assumed independent of $\{C_t\}$, which also satisfy $\mathbb{E}(M_t) = 1$. Consequently, the parameter μ stands for the unconditional mean of both U_t and y_t. As it will be apparent from the definitions of the components C_t and M_t in Sections 2.1 and 2.2, respectively, the parameter vector in the FHMM model, denoted by Θ, includes nine parameters: $\Theta = \{\mu, \alpha, p, c_1, \theta_c, q, m_1, \theta_m, \sigma^2\}$.

2.1 The latent component C_t

Let $\{C_t^{(i)}\}$, $i = 1, \ldots, N$, be N independent two-state Markov chains that share the same 2×2 transition probability matrix (t.p.m.)

$$\boldsymbol{P} = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix},$$

where $p \in (0, 1)$. Assume further that $C_t^{(i)} \in \{c_i, 1\}$, where $c_1 > 1$ and $c_i = (1 - \theta_c) + \theta_c c_{i-1}$ for $i = 2, \ldots, N$ and $\theta_c \in [0, 1]$. Note that this implies a hierarchical structure on the underlying state spaces because $c_1 \geq c_2 \geq \ldots \geq c_N \geq 1$.

The component C_t is constructed as a product of these two-state Markov chains as follows:

$$C_t = c_0 \prod_{i=1}^{N} C_t^{(i)},$$

where $c_0 = \left[\prod_{i=1}^{N} (1 + \theta_c^{i-1}(c_1 - 1)/2)\right]^{-1}$ is a normalizing constant ensuring that $\mathbb{E}(C_t) = 1$. As the estimated value of p is generally very close to 1 in empirical applications, this component captures changes to the conditional mean that have a long-lasting impact.

2.2 The latent component M_t

The component M_t corresponds to a discrete random variable with probability mass function

$$\Pr\left(M_t = m_0 \cdot m_i\right) = \begin{cases} q(N-1)^{-1}, & \text{if } i = 1, \dots, N-1, \\ 1-q, & \text{if } i = N, \end{cases}$$

where $q \in (0,1)$, $m_1 > 1$, $m_i = (1-\theta_m)+\theta_m m_{i-1}$ for $i = 2, \dots, N-1$, and $m_N = 1$. We assume that $\theta_m \in [0,1]$, which once again implies the hierarchical structure $m_1 \geq m_2 \geq \dots \geq m_N = 1$, and use m_0 as a normalizing constant to ensure $\mathbb{E}\left(M_t\right) = 1$, which leads to $m_0 = \left[1 + q\frac{(m_1-1)(1-\theta_m^{N-1})}{(N-1)(1-\theta_m)}\right]^{-1}$. This component adds flexibility to the model because it serves to generate non-persistent moves in the conditional mean.

2.3 Markov chain structure of U_t

Similarly to Augustyniak et al. (2018), it can be shown that the process $\{U_t\}$ corresponds to a first-order Markov chain on the state space of $N \times 2^N$ elements defined by $\mathcal{X}_U = \mathcal{X}_M \otimes (\mu + \alpha(\mathcal{X}_C - 1))$, where $\mathcal{X}_M = m_0 \cdot \{m_1, m_2, \dots, m_N\}$ and $\mathcal{X}_C = c_0 \cdot \{c_1, 1\} \otimes \{c_2, 1\} \otimes \cdots \otimes \{c_N, 1\}$. Its $\left(N \cdot 2^N\right) \times \left(N \cdot 2^N\right)$ t.p.m. is given by

$$\boldsymbol{P}^{\otimes N} \otimes \mathbf{1}_N \left(\underbrace{\frac{q}{N-1}, \quad \dots \quad, \frac{q}{N-1}}_{(N-1) \text{ terms}}, \quad 1-q \right)', \tag{2}$$

where $\boldsymbol{P}^{\otimes N}$ is the Nth Kronecker power of $\boldsymbol{P}$, and $\mathbf{1}_N$ is used to denote the N-dimensional column vector of ones. The asymptotic rate of convergence of this t.p.m. is equal to $O(k^{N-1}|2p-1|^k)$, as $k \to \infty$ (see Augustyniak et al., 2018, Theorem 1). Consequently, the convergence is slowed down either by increasing the number of components N or by having a probability p close to 1 (or 0). As explained by Augustyniak et al. (2018), the process $\{U_t\}$ has the capability to generate a slowly decaying autocorrelation function.

2.4 Autocovariance structure

Proposition 1 provides the autocovariance structure of the FHMM process.

Proposition 1 (Autocovariance structure). *Let* $\gamma = 2p - 1$ *and* $\phi_i = \left(\frac{c_i - 1}{c_i + 1}\right)^2$, *for* $i = 1, \ldots, N$. *For* $k = 1, 2, \ldots$, *we have:*

(i)

$$\operatorname{Cov}(y_t, y_{t+k}) = \operatorname{Cov}(U_t, U_{t+k}),$$

(ii)

$$\operatorname{Cov}(y_t, y_{t+k}) = \alpha^2 \left[\prod_{i=1}^{N} \left(1 + \phi_i \gamma^k\right) - 1\right],$$

(iii)

$$\operatorname{Var}(y_t) = \sigma^2 + \mu^2 \operatorname{Var}(M_t) + \alpha^2 \operatorname{Var}(C_t) + \alpha^2 \operatorname{Var}(M_t) \operatorname{Var}(C_t),$$

where $\operatorname{Var}(C_t) = \prod_{i=1}^{N} (1 + \phi_i) - 1$ *and* $\operatorname{Var}(M_t) = m_0^2 \left(\frac{q}{N-1} \sum_{i=1}^{N-1} m_i^2 + (1 - q)\right) - 1$.

Proof. The proof is given in Appendix A. □

2.5 Relationship to the FHMV model

Since the FHMM process is inspired from the FHMV model of Augustyniak et al. (2018), we highlight the key differences between the two specifications. First, the FHMV process was proposed to model a time series of volatility proxies denoted by $\{x_t\}$, such as realized variances or squared returns, with the following representation:

$$\begin{aligned} x_t &= V_t \eta_t, \\ V_t &= \sigma^2 C_t M_t, \end{aligned} \tag{3}$$

where $\{\eta_t\}$ is a positive i.i.d. innovation process with mean 1, and $\{V_t\}$ corresponds to an unobserved Markov chain that shares the same latent components as $\{U_t\}$. Although the observed variable (y_t or x_t) is a noisy measure of the hidden Markov chain (U_t or V_t) in both the FHMM and FHMV models, the noise term $\{\epsilon_t\}$ in the FHMM process is allowed to take negative values and is specified in an additive rather than multiplicative manner. Moreover, Equations (1) and (3) entail different state space structures for the two Markov chains; V_t is restricted to have positive support whereas U_t has the flexibility to take negative values. However, the transition matrices of these Markov chains are nevertheless identical and correspond to the matrix in Equation (2). Consequently, the autocovariance function in the FHMM model exhibits a similar structure to the one in the FHMV model.

Finally, we remark that Augustyniak et al. (2018) suggested to attach an economic interpretation to the latent components C_t and M_t. This interpretation can be extended to the FHMM model in the sense that these components can represent, respectively, persistent and non-persistent information arrival flow processes that impact the conditional mean U_t with varying degrees of importance.

2.6 Maximum likelihood estimation

As the FHMM process $\{(y_t, U_t)\}$ corresponds to a hidden Markov model, its log-likelihood can be computed using standard filtering techniques (see Hamilton, 1994, chapter 22). However, we note that we can take advantage of the memoryless property of the process $\{M_t\}$ to reduce the computational complexity of the Hamilton filtering algorithm. This algorithm computes the predictive and filtering distributions of the latent component C_t as well as the conditional density of the observed process recursively for $t = 1, \ldots, T$ as follows:

$$\begin{aligned}
\text{Predictive distribution: } p(C_t \mid y_{1:t-1}, \Theta) &= \sum_{C_{t-1} \in \mathcal{X}_C} p(C_t \mid C_{t-1}, \Theta) p(C_{t-1} \mid y_{1:t-1}, \Theta), \\
\text{Observed density: } \quad p(y_t \mid y_{1:t-1}, \Theta) &= \sum_{C_t \in \mathcal{X}_C} p(y_t \mid C_t, \Theta) p(C_t \mid y_{1:t-1}, \Theta), \\
\text{Filtering distribution: } \quad p(C_t \mid y_{1:t}, \Theta) &= \frac{p(y_t \mid C_t, \Theta) p(C_t \mid y_{1:t-1}, \Theta)}{p(y_t \mid y_{1:t-1}, \Theta)},
\end{aligned}$$

where $y_{1:t} = (y_1, \ldots, y_t)$, and $p(y_t \mid C_t, \Theta) = \sum_{M_t \in \mathcal{X}_M} p(y_t \mid C_t, M_t, \Theta) p(M_t \mid \Theta)$. The log-likelihood function is then obtained based on the decomposition $\log p(y_{1:T} \mid \Theta) = \log p(y_1 \mid \Theta) + \sum_{t=2}^{T} \log p(y_t \mid y_{1:t-1}, \Theta)$, and can be maximized numerically using an optimization routine. To initiate the algorithm, we set the state distribution at time $t = 0$, $p(C_0 \mid \Theta)$, to the stationary distribution of the Markov chain $\{C_t\}$.

We note that the number of components N must be specified before maximizing the log-likelihood. Although it is possible to identify the optimal N through repeated estimations, Augustyniak et al. (2018) argue in the context of the FHMV model that it is generally sufficient to consider only one specification with a large N. They explain that the model has the ability to adjust itself, through the parameters c_1, m_1, θ_c and θ_m, and assign very little importance to superfluous components. They illustrate that a value of $N = 10$ generally provides sufficient flexibility in empirical applications.

3 Empirical application

We fit the FHMM process assuming a Gaussian innovation and $N = 10$ by maximum likelihood to 21 monthly time series representing important US macroeconomic variables. These data sets were also studied by Bauwens et al. (2015) and span the period from January, 1959 to September, 2011 (632 monthly observations). Appendix B defines each macroeconomic variable and indicates which transformation was applied on it, if any (such as a transformation to growth rates or first differences).

Table 1 compares the Bayesian information criterion (BIC) of the FHMM model to the BIC of AR(p_{BIC}), MA(q_{BIC}) and ARMA$(p, q)_{\text{BIC}}$ processes, where the lag orders are selected to maximize the BIC. The FHMM model outperforms the best ARMA process on 13 out of 21 series (62 % of the time). This suggests that processes derived from the MSM literature can add value to model the conditional mean of macroeconomic series.

In addition to a competitive fit, the FHMM process displays distinctive dynamics with respect to standard ARMA models. For example, Figure 1 shows the inferred conditional mean over time in the FHMM and AR(p_{BIC}) models for the US inflation growth rate (CPIAUCSL time series). The inferred conditional mean process in the FHMM model is less jittery and follows the level of the time series more steadily. Moreover, it also possesses the flexibility to catch occasional spikes in the time series through the independent component M_t. In general, the conditional mean process in the FHMM model is more robust to outliers and is less affected by noise because to induce a persistent change in the conditional mean, one of the N components comprised in C_t must be activated. However, if the shock is actually short-lived such a regime change will be costly in terms of likelihood, and as a result the component C_t will move only when it detects a long-lasting

Table 1 – Fit results on 21 monthly US macroeconomic time series.

Acronym	FHMM	AR(p_{BIC})	p_{BIC}	MA(q_{BIC})	q_{BIC}	ARMA$(p,q)_{\text{BIC}}$	$(p,q)_{\text{BIC}}$	Δ best model
CPIAUCSL	**45.9**	−14.0	12	−40.1	5	0.1	2, 1	45.8
FEDFUNDS	**980.1**	959.3	14	764.0	5	968.9	5, 5	11.3
BOGNONBR	**1981.1**	107.6	15	136.0	5	142.6	15, 4	1838.4
M2SL	**−53.4**	−127.9	5	−138.0	3	−128.5	1, 4	74.5
INDPRO	**−691.8**	−746.4	2	−746.6	4	−744.6	1, 1	52.8
UTL11	588.7	801.8	3	559.2	5	**803.6**	2, 1	−215.0
UNRATE	1582.5	1653.4	5	1311.3	5	**1657.8**	3, 1	−75.3
HOUST	686.2	710.0	2	527.7	5	**713.1**	2, 2	−27.0
PPIFCG	**−599.0**	−659.9	2	−662.9	3	−656.6	1, 1	57.6
AHEMAN	−525.7	**−405.0**	12	−525.8	3	−406.2	12, 1	−120.7
M1SL	**−492.0**	−611.2	6	−628.2	3	−597.5	4, 4	105.5
PMCP	977.0	**1026.3**	3	965.3	5	1025.7	4, 1	−49.3
SP500	**−222.7**	−244.7	1	−242.5	1	−243.8	2, 2	19.8
GS10	1203.0	1322.0	3	1020.8	5	**1324.0**	1, 1	−121.0
EXUSUK	**213.7**	99.2	3	104.6	1	106.3	2, 3	107.5
PAYEMS	**248.0**	186.9	3	163.9	5	188.1	2, 1	59.9
NAPMNOI	1147.3	**1148.1**	1	1118.3	5	1144.9	1, 1	−0.8
TB3MS	**1124.2**	1086.5	10	864.3	5	1088.1	10, 2	36.1
BUSLOANS	**−569.6**	−593.1	3	−626.5	4	−583.1	4, 3	13.5
TOTALSL	**−185.0**	−227.1	4	−278.6	5	−222.2	1, 1	37.2
AAA	1371.0	1413.3	3	1095.4	5	**1418.1**	6, 5	−47.1

Note: BIC values of the FHMM model assuming a Gaussian innovation and $N = 10$ with respect to AR(p_{BIC}), MA(q_{BIC}) and ARMA$(p,q)_{\text{BIC}}$ processes, where the lag orders are selected to maximize the BIC. The BIC is defined as follows: $\text{BIC} = \ell - 0.5n\log(T)$, where ℓ is the log-likelihood, n is the number of parameters and T is the length of the time series. The highest values are bolded. "Δ best model" denotes the difference in BIC values between the FHMM model and the best alternative process. A positive value is in favor of the FHMM model.

change.

4 Conclusion

The purpose of this letter was to point out that the MSM model and subsequent inspired processes, although traditionally used for volatility modeling, can be adapted to capture time-variation in the conditional mean of a time series with support on the whole real line. In particular, we developed a new regime-switching process to model macroeconomic time series by extending the FHMV model recently proposed by Augustyniak et al. (2018). On 13 out of the 21 time series considered, this model outperformed the best ARMA model according to the BIC. This is a promising performance for a first attempt to model macroeconomic series with this class of models. In fact, it suggests

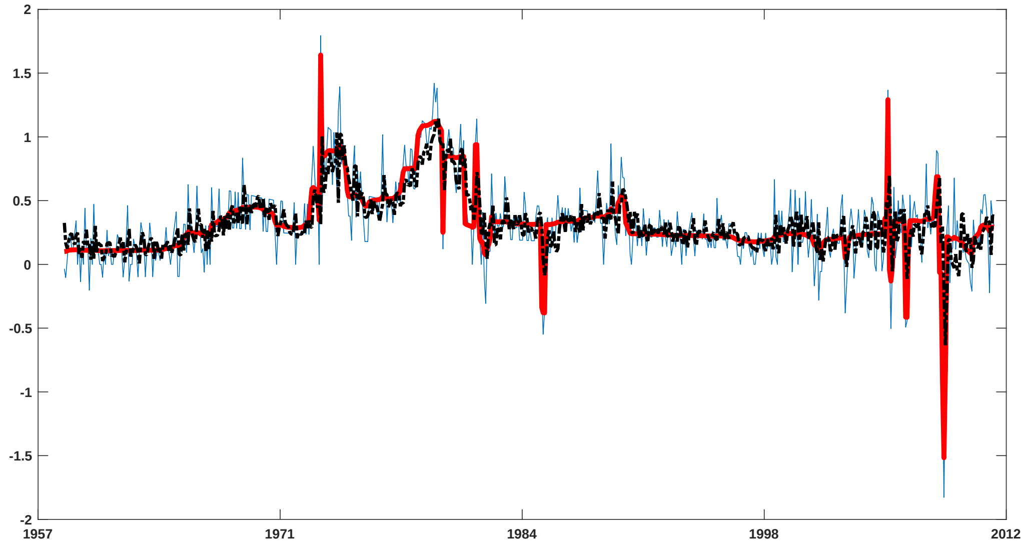

Figure 1 – US inflation growth rate (CPIAUCSL time series): Inferred conditional mean process for the FHMM model (red bolded solid line) and the best AR model with $p_{\text{BIC}} = 12$ (black bolded dashed line). The US inflation growth rate is displayed in the blue solid line.

that such processes should not only be confined to modeling positive processes, such as volatilities, and should warrant additional investigation for dynamically modeling the level of a time series.

A Proof of Proposition 1

We prove Proposition 1.

Proof. Item *(i)* follows from the independence between $\{U_t\}$ and $\{\epsilon_t\}$ since, for $k = 1, 2, \ldots$,

$$\mathbb{E}(y_t y_{t+k}) = \mathbb{E}\left[(U_t + \epsilon_t)(U_{t+k} + \epsilon_{t+k})\right] = \mathbb{E}(U_t U_{t+k}),$$

and hence $\text{Cov}(y_t, y_{t+k}) = \text{Cov}(U_t, U_{t+k})$.

Item *(ii)* is derived from

$$\begin{aligned}\text{Cov}(U_t, U_{t+k}) &= \mathbb{E}(M_t)\mathbb{E}(M_{t+k})\mathbb{E}\left[(\mu + \alpha(C_t - 1))(\mu + \alpha(C_{t+k} - 1))\right] - \mu^2 \\ &= \mu^2 + \alpha^2 \mathbb{E}\left[(C_t - 1)(C_{t+k} - 1)\right] - \mu^2 \\ &= \alpha^2 \left(\mathbb{E}(C_t C_{t+k}) - 1\right),\end{aligned}$$

where $\mathbb{E}(C_t C_{t+k}) = \prod_{i=1}^{N} \left(1 + \phi_i \gamma^k\right)$, as demonstrated in the supplementary appendix of Augustyniak et al. (2018).

Item *(iii)* follows from

$$
\begin{aligned}
\mathrm{Var}(y_t) &= \mathrm{Var}(\epsilon_t) + \mathrm{Var}(U_t) \\
&= \sigma^2 + \mathbb{E}(M_t^2)\mathbb{E}\left[(\mu + \alpha(C_t - 1))^2\right] - \mu^2 \\
&= \sigma^2 + \mathbb{E}(M_t^2)\left(\mu^2 + \alpha^2\left(\mathbb{E}(C_t^2) - 1\right)\right) - \mu^2 \\
&= \sigma^2 + \mu^2\left(\mathbb{E}(M_t^2) - 1\right) + \alpha^2\mathbb{E}(M_t^2)\left(\mathbb{E}(C_t^2) - 1\right) \\
&= \sigma^2 + \mu^2\,\mathrm{Var}(M_t) + \alpha^2\left(\mathrm{Var}(M_t) + 1\right)\mathrm{Var}(C_t) \\
&= \sigma^2 + \mu^2\,\mathrm{Var}(M_t) + \alpha^2\,\mathrm{Var}(C_t) + \alpha^2\,\mathrm{Var}(M_t)\,\mathrm{Var}(C_t).
\end{aligned}
$$

The supplementary appendix of Augustyniak et al. (2018) shows that $\mathbb{E}(C_t^2) = \prod_{i=1}^{N}(1 + \phi_i)$ and $\mathbb{E}(M_t^2) = m_0^2\left(\frac{q}{N-1}\sum_{i=1}^{N-1} m_i^2 + (1-q)\right)$, from which we directly obtain the expressions for $\mathrm{Var}(C_t)$ and $\mathrm{Var}(M_t)$. □

B Macroeconomic series

Table 2 provides the definitions of each macroeconomic variable and indicates the transformations performed on them.

References

Augustyniak, M., Bauwens, L., and Dufays, A. (2018). A new approach to volatility modeling: the factorial hidden Markov volatility model. *Journal of Business & Economic Statistics*. In press, DOI: 10.1080/07350015.2017.1415910.

Bauwens, L., Koop, G., Korobilis, D., and Rombouts, J. V. K. (2015). The contribution of structural break models to forecasting macroeconomic series. *Journal of Applied Econometrics*, 30(4):596–620.

Table 2 – Description of the monthly US macroeconomic time series.

#	Acronym	Trans.	Definition
1	CPIAUCSL	5	Consumer price index: all items.
2	FEDFUNDS	1	Effective federal funds rate.
3	BOGNONBR	2	Non-borrowed reserves of depository institutions.
4	M2SL	5	M2 Money stock.
5	INDPRO	5	Industrial production index.
6	UTL11	1	Capacity utilization: manufacturing.
7	UNRATE	1	Civilian unemployment rate.
8	HOUST	4	Housing starts: new privately owned housing units started.
9	PPIFCG	5	Producer price index: all commodities.
10	AHEMAN	5	Average hourly earnings: manufacturing.
11	M1SL	5	M1 money stock.
12	PMCP	1	NAPM commodity prices index.
13	SP500	5	S&P 500 index.
14	GS10	1	10-year treasury constant maturity rate.
15	EXUSUK	5	US/UK foreign exchange rate.
16	PAYEMS	5	Total nonfarm payrolls: all employees.
17	NAPMNOI	1	ISM manufacturing: new orders index.
18	TB3MS	1	3-month Treasury bill: secondary market rate.
19	BUSLOANS	5	Commercial and industrial loans at all commercial banks.
20	TOTALSL	5	Total consumer credit outstanding.
21	AAA	1	Moody's seasoned Aaa corporate bond yield.

Note: Trans. (transformation applied to original series): 1=none; 2=first difference; 4=log; 5=first difference of logged variable. Data source: St Louis FRED database.

Calvet, L. E. and Fisher, A. J. (2004). How to forecast long-run volatility: Regime switching and the estimation of multifractal processes. *Journal of Financial Econometrics*, 2(1):49–83.

Chen, F., Diebold, F. X., and Schorfheide, F. (2013). A Markov-switching multifractal inter-trade duration model, with application to US equities. *Journal of Econometrics*, 177(2):320–342.

Fleming, J. and Kirby, C. (2013). Component-driven regime-switching volatility. *Journal of Financial Econometrics*, 11(2):263–301.

Hamilton, J. D. (1994). *Time series analysis*. Princeton University Press, New Jersey.

Liu, R., Matteo, T. D., and Lux, T. (2007). True and apparent scaling: The proximity of the Markov-switching multifractal model to long-range dependence. *Physica A: Statistical Mechan-*

ics and its Applications, 383(1):35–42. Econophysics Colloquium 2006 and Third Bonzenfreies Colloquium.

Lux, T. and Kaizoji, T. (2007). Forecasting volatility and volume in the Tokyo stock market: long memory, fractality and regime switching. *Journal of Economic Dynamics & Control*, 31(6):1808–1843.